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Topological Geometrodynamics. III. Quantum Theory

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The description of 3-space as a spacelike 3-surface of the space $H = M^4 \times CP_2$ (product of Minkowski space and two-dimensional complex projective space CP_2) and the idea that particles correspond to 3-surfaces of finite size in H are the basic ingredients of topological geometrodynamics, an attempt to a geometry-based unification of the fundamental interactions. The observations that the Schrödinger equation can be derived from a variational principle and that the existence of a unitary S matrix follows from the phase symmetry of this action lead to the idea that quantum TGD should be derivable from a quadratic phase symmetric variational principle in the space SH, consisting of the spacelike 3-surfaces of H. In this paper a formal realization of this idea is proposed. First, the space SH is endowed with the necessary geometric structures (metric, vielbein, and spinor structures) induced from the corresponding structures of the space H. Second, the concepts of the scalar super field in SH (both fermions and bosons should be describable by the same probability amplitude) and of super d'Alambertian are defined. It is shown that the requirement of a maximal symmetry leads to a unique CP-breaking super d'Alambertian and thus to a unique theory "predicting everything." Finally, a formal expression for the S matrix of the theory is derived.

1. INTRODUCTION

Topological geometrodynamics (TGD) is a geometry-based attempt to unify the fundamental interactions based on the idea that classical space-time can be regarded as a submanifold of some higher-dimensional space H (Pitkänen, 1981, 1983). Once the postulate about representability as a submanifold is accepted, one is led rather naturally to the following scenario.

(1) The concepts of particle and 3-space generalize and, in a certain sense, are unified. Particles (in a very general sense of the word) are identified as spacelike 3-surfaces of H so that a topological classification of particles and particle reactions emerges. Classical 3-space with particles is identified as a topologically trivial 3-surface to which the particlelike 3-surfaces are "glued."

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(2) The natural requirement that isometries of the space H are symmetries of the theory leads to the identification of the space H as the Cartesian product $M^4 \times CP_2$ of Minkowski space and of the space CP_2 the complex projective space (Eguchi et al., 1980; Gibbons and Pope, 1978). The isometry group of the space CP_2 is identified as a color group. Thus one can identify color gravitational interactions as interactions coupling to the isometry charges of the space H.

(3) The so-called induction procedure allows to define gauge potentials on the submanifolds of H as field quantities induced from the spinor connection of the space H. It turns out that these gauge potentials can be identified as electroweak gauge potentials.

(4) The geometrization of the spectroscopy is achieved. The choice explains the quantum numbers associated with a single-particle family and family replication phenomenon has a natrual topological explanation. One can imagine several dynamical scenarios in which either leptons or quarks or both appear as elementary fermions. In the simplest dynamical scenario found thus far leptons are the only elementary fermions and we regard this scenario as the most feasible one.

Concerning the form of the quantum dynamics based on this general framework perhaps the most important achievement hitherto is the realization that the formulation of the theory, whatever it may be, should be free of arbitrary physically relevant parameters. Thus all dimensional parameters (gravitational coupling, masses, ...) should be related to the length scale of CP_2 by some predictable dimensionless numbers (scale invariance is broken by the curvature of CP_2). Also dimensionless couplings should be predictions of the theory.

The use of the conventional quantization methods to construct a quantum theory, which "Predicts everything," is highly questionable since these methods typically describe interactions as nonlinearities in the action defining the theory so that various coupling constants are arbitrary parameters at least at the classical level.

Indeed it must be admitted that the functional integral method has generated only semiclassical arguments but not a calculable quantum theory; even a formal proof for the existence of a unitary S matrix is missing. A probable reason to this circumstance is that the concept of the functional integral makes sense only as a perturbative treatment of conventional field theories.

These experiences suggest that the difficulties are not only technical but that the underlying philosophy is in some way wrong. Indeed, one can argue that the idea of constructing quantum theory by first postulating classical action and then quantizing it through some more or less unique rules is wrong.

The quantization philosophy adopted in this paper is in concise form "Do not quantize!". The new line of thought is based on the following observations:

(1) The ordinary Schrödinger equation is obtained from a variation principle (Appendix) and the associated action is quadratic with respect to the probability amplitude.

(2) The existence of a Hilbert space scalar product and of a unitary S matrix results from the conservation law of probability, which in turn follows from the phase symmetry of the quadratic action.

From these observations we abstract our basic receipe of a quantum theory:

(1) Assume the existence of a configuration space SH endowed with metric structure (spacelike submanifolds of H).

(2) Postulate a variational principle for the probability amplitudes defined in SH with the property that the associated action is quadratic with respect to the probability amplitudes and invariant under phase symmetries.

This idea is the only completely new ingredient in our approach. The action principle is fixed to high degree by requiring the following:

(3) The probability amplitudes are geometric objects and the action is defined by a Lagrangian density invariant under the coordinate transformations of the space SH.

More concretely, the quantum equations of motion should correspond to "massless" (no free physically relevant parameters) d'Alambert type equations in configuration space endowed with a metric.

Furthermore it is natural to assume the following:

(4) The geometry of the configuration space (spacelike 3-submanifolds of the space H) is induced from the geometry of H (metric, Riemannian connection, and spinor structure).

Since the state functionals must be capable of describing both bosons and fermions and states of arbitrarily high fermion number it is natural to postulate the following:

(5) A state functional is a Grassmann algebra valued "scalar superfield" (Hawking and Pope, 1978; Volkov, 1973; Wess and Zumino, 1974; Stelle, 1983). The generators of the Grassmann algebra ("theta parameters") in a given point of the configuration space (3-submanifold of H) are in one-to-one correspondence with a complete spinor field basis defined in the corresponding 3-manifold of H.

The success of this program would mean that a linear theory containing no physically relevant free parameters would produce a nontrivial S matrix. In order to understand how this is possible consider the structure of the configuration space SH. SH is obtained by glueing together spaces $SH(t, n)$ corresponding to 3-manifolds with a given number of components ("particle number") with given topologies t. The points common to $SH(t1, m)$ and $SH(t2, m)$ correspond to surfaces topologically intermediate between manifold topologies $t1$ and $t2$ and are singular as manifolds.

Consider now a state functional corresponding to a well-defined particle number n and thus restricted to $SH(n)$. The uncertainty principle in $SH(!)$ implies that this state cannot be stationary but begins to disperse to other parts of SH with different particle numbers. Clearly, this process leads to occurrence of particle reactions.

The plan of the paper is as follows:

In Section 3 we discuss the problem of defining SH as manifold and propose an induction procedure to obtain metric and spinor structures in SH from those of the space H.

In Section 4 we consider the definition of a scalar superfield in SH. We construct the formalism in finite-dimensional case first and then perform a generalization to the case of SH. It turns out that the requirement of maximal symmetry leads to an essentially unique super d'Alambertian characterized by the so-called Abelian super gauge invariance.

In Section 5 we show that in case of $H = M^4 \times CP_2$ the Kähler structure of CP_2 makes possible a CP-breaking term in the Super d'Alambertian and that the requirement of maximal symmetry fixes this term uniquely.

Section 6 is devoted to the construction of the S matrix. The so-called bare states are defined as state functionals restricted to a subset $SH(t)$ or SH consisting of surfaces with fixed manifold topology t. Stationary states are defined as continuations of the bare state functionals to state functionals in the whole SH. A formal solution of the continuity conditions is derived and conditions guaranteeing the uniqueness of the continuation are deduced. Finally, an explicit expression for S matrix as a unitary matrix transforming bare states to stationary states is derived.

2. NOTATION

Symbol	Meaning
$H = V \times S$	imbedding space, which is Cartesian product of V and S
M^4/M^4_+	Minkowski space/light cone of Minkowski space
CP_2	complex projective space of complex dimension 2
$h^k/m^k/s^k$	coordinates for space $H/M^4/S$
ξ^k, $\bar{\xi}^k$, $k = 1, 2$	complex coordinates for CP_2
x^α/ξ^μ	coordinates for the interior/boundary component of a submanifold X^3
$h^k_{1\alpha}/m^k_{1\alpha}/s^k_{1\alpha}$	partial derivatives of the coordinate variables of $H/M^4/S$ with respect to the coordinate variables of X^3

$h_{kl}/m_{kl}/s_{kl}$	components of the metric tensor for $H/M^4/S$
e_k^A	components of the vielbein in H
$V_k/B_k/A_k$	components of vielbein connection/Kähler potential in CP_2/spinor connection in H
$V_{kl}/J_{kl}/F_{kl}$	curvature form of vielbein connection/Kähler form/curvature form of spinor connection
$g_{\alpha\beta}=h_{kl}h^k_{1\alpha}h^l_{1\beta}$	induced metric in X^3
$V_\alpha/B_\alpha/A_\alpha$	induced vielbein connection/Kähler potential/spinor connection in X^3
$H^k_{\alpha\beta}=D_\beta h^k_{1\beta}$	second fundamental form for X^3
$H^k=g^{\alpha\beta}H^k_{\alpha\beta}$	trace of the second fundamental form
Γ_k	gamma matrices for the space H
$\Gamma_\alpha=\Gamma_k h^k_{1\alpha}$	gamma matrices for X^3
γ_A	flat space gamma matrices
$\Sigma_{AB}=[\gamma_A,\gamma_B]/2$	flat space sigma matrices
$\tilde{\Gamma}_k$	modified gamma matrices of the space H
D_k/D_α	covariant derivative in H/X^3
X^n	n-dimensional submanifold of H
Int X^3	interior of X^3
$\delta_i X^3$	ith boundary component of X^3
X	common symbol for Int X^3 and $\delta_i X^3$
$T_I/N_I/T_\delta/N_\delta$	Tangent/normal space of Int $X^3/\delta X^3$
P_I^{kl}	Projection operator to N_I
P_δ^{kl}	Projection operator to $T_I\cap N_\delta$
SH	space of the spacelike 3-surfaces of H
$SH(t)$	set of 3-submanifolds of H with topology t
$SH(t,n)$	set of 3-submanifolds of H having n components with fixed topologies
$SH(t1,t2)$	set of 3-surfaces having topology intermediate between the manifold topologies $t1$ and $t2$
$H_s(X)$	Hilbert space spanned by scalar functions in X
$H_v(X)$	Hilbert space spanned by H-vector fields orthogonal to the surface X
$H_{sp}(X)$	Hilbert space spanned by the H-spinor fields in X ($X=$ Int $X^3/\delta_i X^3$)
$H_i(X^3)$	direct sum of Hilbert spaces $H_i(X)$ ($i=S, V, sp$)
$\{\phi^m_x\}/\{f^m_x\}/\{\psi^m_x\}$	complete basis in the Hilbert space $H_s(X)/H_v(x)/H_{sp}(X)$
$K=(k,x)/L=(1,y)$	shorthand notation for the indices of the various tensor quantities defined in $SH(1)$
$H_{KL}\equiv H_{kl}(x,y)$	metric tensor for $SH(t,1)$.
$e_k^{\bar{A}}$	X part of vielbein of $SH(1)$ ($X=$ Int $X^3/\delta_i X^3$)

$\bar{A}=(A, m)$	shorthand notation for vielbein index; A refers to the components of H vielbein; m refers to an element of complete basis of scalar fields in X
$V_k/B_k/A_k$	vielbein connection/Kähler potential/spinor connection in $SH(1)$
D_K	spinor covariant derivative in $SH(1)$
$\Gamma_{\underline{K}}$	gamma matrices in $SH(1)$
$\gamma^{\bar{A}}/\Sigma^{\bar{A}\bar{B}}$	flat space gamma/sigma matrices in $SH(1)$
$\tilde{\Gamma}_K$	modified gamma matrices in $SH(1)$
$GR(X^3)=\bigoplus_X GR(X)$	spinor Grassmann algebra of X^3 representable as a direct sum of Grassmann algebras $GR(X)$ ($X=\text{Int}\, X^3$, $\delta_i X^3$)
$\theta_X^\alpha, \bar{\theta}_X^\alpha$	theta parameters spanning spinor Grassman algebra $GR(X)$.
$S/S_{M,\bar{N}}$	super scalar field/component of the super field in shorthand notation
$D_\alpha^\varepsilon/\bar{D}_\alpha^\varepsilon$	super covariant derivatives
$\boxtimes$	super d'Alambertian
$S^m(t)$	bare state functional corresponding to given 3-manifold topology t
$S_s^m(t)$	stationary state functional obtained as a continuation of $S^m(t)$

3. ABOUT THE STRUCTURE OF THE CONFIGUATION SPACE

The general structure of the configuration space will be the topic of this section. We shall discuss the topological structure of the configuration space and perform the generalization of the finite dimensional differential geometry to "functional differential geometry" in space SH by defining metric, vielbein, and spinor structures in SH as structures induced from the space H.

3.1. The Topological Structure of the Space *SH*

In this section we define the concept of the configuration space SH as the set of the spacelike 3-surfaces of the space H. A definition of the "allowed surface" is proposed; various physically interesting subsets of SH are defined and finally a representation of the space SH as a union of these subsets is given.

3.1.1. The Concept of the Surface

First it should be emphasized that the definition of the configuration space differs from that used in quantum mechanics since we do not pose the restriction that the surfaces are in, say, a spacelike hyperplane of *H*. We only require that the connected components of the 3-surfaces are spacelike, that is, the induced metric on the surface is everywhere spacelike.

Concerning the definition of the surface concept the following remarks should be made.

(i) By submanifold we mean an imbedding of a manifold to *H*. The manifold can have several connected components and each component can have several boundary components.

(ii) The concept of the surface is more general than that of a submanifold. What is required that surface is locally manifold at almost all its points. Thus surface can have self-intersections and pinches, etc. In Pitkänen (1981, 1983) we have discussed the description of the particles as submanifolds of *H*.

(iii) The singular submanifolds of *H*, which correspond to topology changing transitions of 3-manifolds, have a central role in the description of the interactions in TGD. These surfaces are intermediate between two manifold topologies. Examples of the singular manifolds are the surfaces intermediate between (1) sphere and a union of two disjoint spheres, (2) torus and sphere, and (3) sphere with one and two holes, respectively.

We have proposed a rough classification of the various topology changes and discussed the description of various particle reactions in terms of the intermediate topologies (Pitkänen, 1981, 1983). It is natural to require that all allowed singular surfaces have topologies intermediate between two manifold topologies.

One can also pose restrictions on the nature of the topology change. A possible restriction of this kind is that the topology change is localized so that the intermediate surface fails to be a manifold in a finite set of points. This requirement would exclude the so-called $\#_B$ vertex introduced in earlier publications (Pitkänen, 1981, 1983).

3.1.2. Various Subsets of SH

On physical grounds one expects that the subsets of *SH* consisting of submanifolds of *SH* with a fixed topology are of same dimension than *SH* itself and that the subsets consisting of singular manifolds in some sense form a "measure zero" subset of *SH*. In the following we give a more precise formulation of these intuitive notions.

The idea that submanifolds with a fixed topology form a subset of *SH* having the same dimension as *SH* derives from their property of being open sets of *SH*. That these sets are open following from the invariance of the property of being submanifold with a given topology under small deformations of the surface.

The following subsets of SH are obviously open:

$SH(t)$ = set of manifolds with a given topology t
$SH(t, n)$ = set of manifolds having n disjoint components with given topologies.
$SH(n)$ = set of manifolds with n components and with arbitrary topologies
$SH(t, n, m_1, \ldots, m_n)$ = set of n-component 3-manifolds, the kth component having m_k boundary components.

The subsets of SH consisting of surfaces, which are singular as submanifolds of H are not open since an arbitrary small deformation can lead to a final state, which corresponds to manifold topology (perform a small deformation near the "reaction vertex"). We shall denote by the symbol $SH(t1, t2)$ the set of 3-surfaces having topologies intermediate between the manifold topologies $t1$ and $t2$.

Since the singular manifolds are in a certain sense limiting cases of regular manifolds, the sets $SH(t, t2)$ for a given t must belong to the compactification of the set $SH(t)$, which we shall denote by the symbol $\overline{SH}(t)$; in other words the boundary of $\overline{SH}(t)$ is given by

$$\delta\overline{SH}(t) = \bigcup_{t1} SH(t, t1) \tag{1a}$$

It is also clear that the sets $SH(t1, t2)$ can be represented as the intersections of the compactifications $SH(t1)$ and $SH(t2)$:

$$SH(t1, t2) = \overline{SH}(t1) \cap \overline{SH}(t2) \tag{1b}$$

With these identifications one can regard SH as a manifold obtained by glueing the sets $SH(t)$ together along their common boundaries [the subset $SH(t1, t2)$ and $SH(t2, t1)$ are identified in the union of $\overline{SH(t)}$]:

$$SH = \bigcup_{t} \overline{SH}(t) \tag{1c}$$

There is a relationship between the sets $SH(1)$ and $SH(n)$: namely, to each element of $SH(n)$ (disjoint union of n manifolds) there corresponds a unique element in the set $SH(1) \times SH(1) \times \cdots \times SH(1)/S_n$, where n is the number of the factors in the product and S_n is the permutation group of n objects.

The division by S_n means identification of all elements obtained from $s_1 \times s_2 \times \cdots \times s_n$ by permuting the surfaces s_k, $k = 1, \ldots, n$. The identification of the different permutations derives from the communtativity of the set theoretic union.

This kind of relationship between one-particle and many-particle configuration spaces is encountered already in ordinary quantum mechanical treatment of a many particle system. The euclidian 3-space E is the configuration space for a point particle and the space $E \times E \times \cdots \times E / S_n$ is the configuration space for n particles. In practice one can use the configuration space $E \times E \times \cdots \times E$ and assume the state functions be completely symmetric/completely antisymmetric under particle exchanges. The completely antisymmetric state functions correspond to double-valued state functions in real configuration space.

The above-mentioned correspondence is important since the assumption about the symmetry properties of the state functions reduces the geometrization of the space SH to the geometrization of $SH(1)$ since the geometry of $SH(n)$ is essentially that of the Cartesian product of n $SH(1)$:s locally.

3.2. Metric Structure in SH

3.2.1. General Considerations

The above-mentioned relationship between $SH(n)$ and the Cartesian product of n $SH(1)$:s reduces the geometrization of SH to that of $SH(1)$. The basic philosophy in the following developments is that the geometry of SH is induced from that of the space H. We shall derive the metric tensor of $SH(1)$ from the expression for the line element at a given point of $SH(1)$ (submanifold of H). The expression for the line element is derived by studying small deformations of a given 3-manifold.

Because the manifolds considered in general have arbitrary number of boundary components there are two basic contributions to the metric corresponding to the deformations changing the interior and boundary components of the 3-manifold, respectively.

Since the tangential part of the interior deformation does not change the surface but gives rise only to a coordinate change it is clear that only the normal space (denoted by N_I) projection of the deformation contributes to the line element.

In a similar way, only the projection of the boundary deformation to the normal space N_δ of the boundary contributes to the line element. In addition, in order to avoid double counting, the relevant part of the deformation must belong to the tangent space T_I of Int X^3. Thus only the projection of the boundary deformation to the one-dimensional dimensional space $T_I \cap N_\delta$ contributes to the boundary part of the line element.

3.2.2. Derivation of the Line Element

To derive the interior contribution to the line element at a given point of $SH(1)$ we make the following assumptions:

(a) The line element is integral over the contributions coming from infinitesimal coordinate volumes d^3x.

(b) The contribution of an infinitesimal volume element is the orthogonal distance between the infinitesimal surface corresponding to the volume element and the infinitesimal surface obtained by performing a displacement dh^k.

Dividing X^3 in small cells with coordinate volumes $DV_n = d^3x_n$ and using the assumptions one obtains the following expression for the line element:

$$DS^2 = \lim \sum_n P_{kl}(dh^k\, dh^l / dV_n)\, dV_n \tag{2a}$$

$$P^{kl} = h^{kl} - g^{\alpha\beta} h^k_{1\alpha} h^l_{1\beta} \tag{2b}$$

The operator P^{kl} is the projection operator to the normal space N_I of Int X^3.

Clearly, the end result is meaningful only if it is sensible to speak about the density of the line element. This in turn implies that the limits

$$DH^k(x) = \lim dh^k(x_n)/(dV_n)^{1/2} \tag{3}$$

must exist.

We shall call the quantities DH^k deformation half-densities (analogous to the coordinate space wave functions in quantum mechanics).

It is easy to verify that the end result is invariant under the coordinate transformations of X^3 (by dividing deformation half-densities with the fourth root of the metric determinant and by multiplying the volume element by square root of the metric determinant one obtains a manifestly coordinate invariant expression for the line element).

The contribution of a given boundary component to the line element has the same formal structure as the interior contribution:

$$DS^2_\delta = \int P^\delta_{kl} DH^k_\delta DH^l_\delta \, d^2\xi \tag{4a}$$

$$P^{kl}_\delta = g^{\alpha\beta}_I h^k_{1\alpha} h^l_{1\beta} - g^{\mu\nu}_\delta h^k_{1\mu} h_{1\nu} \tag{4b}$$

The operator P^δ_{kl}projects to the space $T_I \cap N_\delta$. The subscripts I and δ refer to interior and boundary metric, respectively; the subscripts (α, β) and (μ, ν) refer to the coordinate variables of interior and boundary component respectively.

3.2.3. *Identification of the Metric Tensor*

Expressing the deformation as a sum of the interior and boundary parts

$$DH^k = DH^k_I + DH^k_\delta \tag{5}$$

and assuming that the imbedding of the ith boundary component is given in the form

$$x^{\alpha} = f^{\alpha}(\xi^{\mu}) \tag{6}$$

where ξ^{μ} denote the coordinate variables associated with the boundary component, one obtains the following expression for the line element

$$DS^2 = \int H_{kl}(x, y) DH^k(x) DH^l(y)\, d^3x\, d^3y \tag{7a}$$

$$H_{kl}(x, y) = \delta^3(x, y) P^I_{kl} + \sum_{\delta} \int \delta^3(x, f(\xi)) \delta^3(y, f(\xi)) P^{\delta}_{kl}\, d^2\xi \tag{7b}$$

The quantity $H_{kl}(x, y)$ is analogous to the metric tensor defining the metric for finite-dimensional spaces. Contraction and raising of indices can be performed using the covariant form of the metric obtained by raising the indices of the projection operators using the metric of the space H.

Tensor $H_{kl}(x, y)$ differs from the ordinary metric tensor since the mixed tensor is not equal to unit matrix. In fact this tensor has no inverse since it annihilates the deformations tangential to the surface (which do not change the surface). Irrespective of this quantity is very useful since one can construct the differential geometry of SH using this quantity and without introducing any explicit coordinates for the space SH.

In the following we shall use the shorthand notation H_{KL} for the metric tensor, indices K and L referring to pairs (k, x) and (l, y), respectively. The formal summations over the index K are understood as integrations over the coordinate variables and summations over the index k.

3.3. Vielbein Structure in *SH*

Let us first state the definition of the vielbein in the finite-dimensional case (Eguchi et al., 1980).

A vielbein at a given point of a finite-dimensional space H is a complete set of tangent vectors at each point of finite-dimensional space with the property the scalar products with respect to flat tangent space metric generate the components of the metric tensor.

In order to obtain a globally defined vielbein one must assume the orientability of the manifold H. In performing the generalization we shall assume that the space SH is also orientable.

Instead of the straightforward formal generalization of this definition we define the vielbein so that it generates the metric tensor H_{KL}.

(a) Let X denote either interior or one of the boundary components of X^3 and let $\{f^m_x\}$ be complete sets of real-valued functions defined in X

satisfying the completeness and orthogonality relations

$$\sum_m f_x^m(x) f_x^m(y) = \delta(x, y) \tag{8a}$$

$$\int f_x^m(x) f_x^n(x)\, dV = \delta^{m,n} \tag{8b}$$

Observe that the functional derivatives of the functions f_x^m vanish because they have not dependence on the coordinate variables h^k.

(b) Define the vielbein associated with the interior of X^3 through the formula

$$E_K^{\bar{A}} \equiv E_k^{A,m}(x) = e_l^A P_k^l f_l^m(x) \tag{9}$$

(here the shorthand notations $\bar{A} = (A, m)$ and $K = (k, x)$ are used).

(c) Define the vielbein associated with the ith boundary component via the formula

$$E_K^{\bar{A}} = E_k^{A,m}(x) = \int \delta(x, f(\xi)) e_l^A P_{\delta k}^l f_\delta^m \, d^2\xi \tag{10}$$

It is easy to verify that these vielbeins generate the metric tensor in the sense that the equations

$$\sum_{\bar{A}} E_K^{\bar{A}} E_L^{\bar{A}} = H_{KL} \tag{11}$$

hold true.

3.4. Induction Procedure for Connections

3.4.1. General Considerations

In finite-dimensional case the defining condition of the metric (vielbein) connection is the covariant constancy of the metric (vielbein) with respect to the covariant derivative defined by the connection (Eguchi et al., 1980). This condition is a natural defining condition also in the infinite-dimensional case.

For practical reasons we define metric and vielbein connections by requiring the covariant constancy of the metric tensor H_{KL} and the associated vielbein.

The covariant constancy conditions are formally same as in the finite-dimensional case and given by

$$H_M^R H_{KL|R} + \left\{ \begin{matrix} R \\ KM \end{matrix} \right\} H_{RL} + \left\{ \begin{matrix} R \\ LM \end{matrix} \right\} H_{RK} = 0 \tag{12}$$

and by

$$D_L E_K^{\bar{A}} + \left\{ \begin{matrix} M \\ LK \end{matrix} \right\} E_M^{\bar{A}} + V_{L\bar{B}}^{\bar{A}} E_K^{\bar{B}} = 0 \tag{13}$$

for the metric and vielbein, respectively.

Also the solutions to these conditions are formally similar to those of the finite-dimensional case. The Christoffel symbols and the components of the vielbein connection are given by

$$\left\{ \begin{matrix} K \\ LM \end{matrix} \right\} = H^{KR}(H_{RL|M} + H_{RM|L} - H_{LM|R})/2 \tag{14}$$

and by

$$V_{L\bar{B}}^{\bar{A}} = -E_{K|L}^{\bar{A}} E_{\bar{B}}^{K} - \left\{ \begin{matrix} M \\ LK \end{matrix} \right\} E_M^{\bar{A}} E_{\bar{B}}^{K} \tag{15}$$

respectively.

3.4.2. Evaluation of the Functional Derivatives

In order to evaluate explicit expressions for metric and vielbein connections one must derive formulas for functional derivatives appearing in the formulas defining them. The expressions for the functional derivatives are obtained by studying variations of the metric and vielbein under certain type of deformations of X^3.

It is natural to assume that the allowed deformations of X^3 consist of two parts; the interior part and boundary part. Furthermore, we can assume that interior part of the variation is orthogonal to Int X^3 and boundary parts of the deformations belong to the space $T_I \cap N_\delta$. This means that the variation dh^k can be written as a superposition of the interior and boundary parts

$$\delta h^k = \delta h_I^k + \delta h_\delta^k \tag{16a}$$

$$\delta h_I^k \| N_I \tag{16b}$$

$$\delta h_\delta^k \| T_I \cap N_\delta \tag{16c}$$

It is important to notice that the variations are now densities and not half densities as in the case of the definition of the line element. Boundary variations are also more singular since they have coordinate length dimension -2 as opposed to the interior variations, which have coordinate length dimension -3.

Since we shall need only the functional derivatives of the metric and vielbein we can assume that the quantity under study is local and depends only on the coordinate variables of H and their first derivatives.

In order to derive expressions for the functional derivatives of the quantity $S(x)$ express the variation of this quantity in the form

$$\delta S(x) = \sum_X (\delta S(x)/\delta h^k(z))\delta h^k(z)\, d^3z \tag{17}$$

and identify the coefficients of the deformation densities (not half-densities!) as the partial functional derivatives.

Explicit expressions for the functional derivatives are obtained by directly calculating the variation of the quantity $S(x)$

$$\delta S(x) = A_k \delta h_I^k + B_k^\alpha \delta h_{I|\alpha}^k + \sum_\delta C_k^\mu \delta h_{\delta|\mu}^k \tag{18a}$$

$$A_k = \frac{\partial S}{\partial h^k} \tag{18b}$$

$$B_k^\alpha = \frac{\partial S}{\partial h_{|\alpha}^k} \tag{18c}$$

$$C_k^\mu = \frac{\partial S}{\partial h_{|\mu}^k} \tag{18d}$$

(recall that indices α and μ refer to the coordinate variables of interior and boundary, respectively).

The terms containing the derivatives of the variation can be transformed away by using the fact that in the cases of interest the quantities B_k^α and C_k^μ are parallel to Int X^3 and δX^3, respectively. Thus one can write the variation in the following form:

$$\delta S(x) = L_k S \delta h^k(x) \tag{19a}$$

Here the quantity $L_k S$ is the Lagrangian derivative of S defined by

$$L_k S = A_k - P^l_{Ik} B^\alpha_{l|\alpha} - \sum_\delta P^l_{\delta k} C^\mu_{l|\mu} \tag{19b}$$

From this expression we obtain the formula for the functional derivative of $S(x)$ applicable to the present situation:

$$\delta S(x)/\delta h^k(z) = L_k S(x)\delta^3(x, z) \tag{20}$$

3.4.2. Explicit Expressions for Metric and Vielbein Connections

The explicit formulas for the components of the metric connection are given by

$$\begin{Bmatrix} K \\ LM \end{Bmatrix} = \delta^3(x, y)\delta^3(y, z) A^k_{lm}(x) \tag{21a}$$

$$A^k_{lm}(x) = P^{kr} P^s_l P^t_m (B_{tsr} + B_{srt} - B_{rst})/2 \tag{21b}$$

$$B_{klm} = L_k P_{ml} \tag{21c}$$

$$P_{kl}(x) = P^I_{kl}(x) - \sum_\delta \int \delta^3(x, f(\xi)) P^\delta_{kl}(\xi)\, d^2\xi \tag{21d}$$

The action of the Lagrangian derivative L_k is defined by the formula (19).

The explicit representation of the vielbein connection is given by

$$V^{\bar{A}}_{K\bar{B}} = P^r_k (L_r E^{\bar{A}}_l E^l_{\bar{B}} + A^m_{k,l} E^{\bar{A}}_m E^k_{\bar{B}}) \tag{22}$$

Here a shorthand notation for the components of the vielbein is used [see equations (10)-(11)].

3.5. Spinor Structure in *SH*

In this section we shall define spinor structure in $SH(1)$. The definition of the various concepts involved (spinor field, gamma matrices, spinor connection) is based on the straightforward generalization from the finite-dimensional case (Eguchi et al., 1980).

3.5.1. Generalization of the Spinor Field Concept

In the following we assume that the concept of spinor as a representation of the spin group of the n-dimensional manifold is familiar to the reader. The spinor bundle is defined as a bundle having as fiber the spinor space V associated with space H and as base the manifold H itself.

A spinor field is a section in this bundle, i.e., a map from H to the spinor bundle commuting with the bundle projection. If the bundle is trivial the spinor field is simply a map from H to the spinor space; in general case this representation is possible only locally (in some open neighborhood U of the given point of H).

Given the local representation of the spinor field as a map from H to the spinor space in some open set $U \subset H$, one obtains the component representation of the spinor field by expanding the spinor field in a complete orthonormalized spinor basis introduced at each point of U.

To perform the generalization notice that the point of the space $SH(1)$ corresponds to a spacelike 3-manifold with arbitrary many boundary components. Furthermore, the spinor structure of the space H induces a spinor structure in the interior and on the boundary components of the 3-surface X^3. The definition of the induced spinor structure is considered in earlier papers (Pitkänen, 1981, 1983). Obviously, one can regard the spinor fields on interior and on the boundaries as restrictions of H-spinor fields.

A natural generalization of the spinor space associated with a given point of H is the direct sum of Hilbert spaces $H_{sp}(X)$ spanned by the H-spinor fields associated with Int X^3 and the various boundary components (we shall use the common symbol X for the interior and various boundary components of X^3).

$$H_{sp}(X^3) = \bigoplus_X H_{sp}(X) \tag{23}$$

The scalar product of the spinor fields ψ_1 and ψ_2 in the Hilbert space $H_{sp}(X)$ is defined by the integral

$$(\psi_1, \psi_2) = \int_X \psi^*_{1\alpha}(x)\psi^\alpha_2(x)\, dV \tag{24}$$

The Hilbert space $H_{sp}(X)$ is assumed to have a complete orthonormalized spinor bases $\{\psi^m_x\}$ so that the conditions

$$(\psi^m_x, \psi^n_x) = \delta^{m,n} \tag{25a}$$

$$\sum_m \psi^*_{m\alpha}(x)\psi_{m\beta}(y) = \delta(x, y)\delta_{\alpha,\beta} \tag{25b}$$

are satisfied.

We define spinor field in $SH(1)$ locally as a map, which associates with Int X^3 and each boundary component an H-valued spinor field. In a more careful treatment one should define spinor field in $SH(1)$ as a section in the "bundle" having $SH(1)$ as a base space and the direct sum of the Hilbert spaces $H_{sp}(X)$ as a fiber. The bundle formulation is not expected to be a completely trivial task since the Hilbert associated with topologically nonequivalent 3-manifolds are not expected to be isomorphic [for example, the number of summands in the direct sum representation of $H_{sp}(X^3)$ varies].

A component representation for the spinor field in $SH(1)$ is obtained by expressing the H-spinor fields associated with Int X^3 and various boundary components in the basis $\{\psi^m_X\}$. Thus the components of the $SH(1)$ spinor field are analogous to Fourier coefficients of the ordinary spinor field.

3.5.2. Gamma Matrices in $SH(1)$

We assume that the definition of the gamma matrices for finite-dimensional curved space is familiar to the reader. We shall define the infinite-dimensional gamma matrices by requiring that they generate the metric tensor H_{KL} rather than the proper metric tensor.

Thus the defining anticommutation relations are given by

$$\{\Gamma_K, \Gamma_L\} = 2H_{KL} \tag{26}$$

One can represent gamma matrices using the vielbein coefficeints $E_K^{\bar{A}}$ and flat space gamma matrices $\gamma_{\bar{A}}$ acting in the Hilbert space $H_{sp}(X^3)$:

$$\Gamma_K = E_K^{\bar{A}} \gamma_{\bar{A}} \tag{27}$$

$$\{\gamma_{\bar{A}}, \gamma_{\bar{B}}\} = 2E\delta_{\bar{A},\bar{B}} Id \tag{28}$$

Here the matrix Id is an identity matrix in $H_{sp}(X^3)$. The parameter E equals to ± 1.

The construction of the representation for the flat space gamma matrices $\gamma_{\bar{A}}$ as operators in the space $H_{sp}(X^3)$ is an important task to be performed.

3.5.3. *Spinor Connection in* $SH(1)$

In the finite-dimensional case spinor connection is obtained simply by lifting the vielbein connection with gauge group $SO(n)$ to a connection having as gauge group the spin group Spin(n), the covering group $SO(n)$ (Eguchi et al., 1980).

In practice this means the definition of the spinor covariant derivative via the formula

$$D_k = \partial_k + V_k \tag{29a}$$

$$V_k = V_{kB}^A \Sigma_A^B / 2 \tag{29b}$$

Here the quantities V_{kB}^A are the coefficients of the vielbein connection.

In the case of $H = M^4 \times CP_2$ the life is not so simple. Since CP_2 does not allow conventional spinor structure (the lift of the vielbein connection to spinor connection is not well defined). As explained in Pitkänen (1983), Eguchi et al. (1980), Gibbons and Pope (1978), Hawking and Pope (1978), one can however define a generalized spinor structure in $M^4 \times CP_2$ by adding an additional $U(1)$ gauge potential to the spinor connection. The $U(1)$ potential is an odd multiple of the Kähler potential, which generates the Kähler form in CP_2.

A naive guess is that the spinor connection for $SH(1)$ contains, not only the vielbein part, but also an additional term and that this term is obtained by inducing the Kähler potential to a gauge potential in $SH(1)$.

A possible way to induce the Kähler potential to the configuration space is based on the following requirements. The induced Kähler potential $[B_K = B_k(x)]$ (i) must have parts B_K^X corresponding to interior and boundary components of X^3, (ii) must be proportional to the projection operator $P_\pm$ of P_δ since pure coordinate transformations cannot contribute in parallel translation, and (iii) must change by a gauge transformation in a gauge transformation of the Kähler potential (gauge invariance of the induction procedure).

It is rather straightforward to verify that the expression

$$B_K = P_k^{Il} B_l(h(x)) + \sum_\delta \int \delta^3(x, f(\xi)) P_k^{\delta l} B_l(h(\xi))\, d^2\xi \tag{30}$$

satisfies these requirements.

The resulting expression of the spinor connection is given by

$$A_K = V_K + B_K(n_+ 1_+ + n_- 1_-) \tag{31}$$

Here the odd integers describe the coupling of the Kähler potential to the spinors of definite H chirality. The simplest possible scenario in which leptons are the only elementary fermions corresponds to the choice $(n_+, n_-) = (0, 3)$ as shown in the first paper of the series.

The proposed definition of the induced Kähler potential, although based on natural requirements, is based on a guess. A more rigorous definition of the induced Kähler potential should be based on definition of the induced Kähler structure in the configuration space SH.

3.5.4. *Coordinates for $SH(t, 1)$*

The formalism constructed in previous sections does not use explicit coordination of $SH(1)$. One cannot however totally avoid the introduction of explicit coordinates to the space SH; for instance the definition of the integration measure necessitates the introduction of the proper metric tensor, and thus of an explicit coordination of $SH(t, 1)$. In the following we introduce local coordinates of $SH(t, 1)$ valid in some neighborhood of a given 3-surface X^3 and derive expressions for the metric tensor and related quantities in these coordinates.

A set of coordinates for $SH(t, 1)$ expected to be applicable in some open neighborhood of a point of $SH(t, 1)$ are given by the Hilbert spaces $H_V(X)$ of the H-valued vector fields f_X:

$$f_X^K = f_X^k(x) \tag{32}$$

(a) of Int X^3 ($X = \text{Int } X^3$) orthogonal to Int X (interior degrees of freedom) and (b) of δX^3 ($X = \delta X^3$) orthogonal to δX^3 and parallel to Int X^3 (boundary degrees of freedom).

Let the set $\{f_{Xm}\}$ form orthonormalized complete basis for the space H at the surface X^3

$$\int_X h_{kl} f^k_{Xm} f^l_{Xn} = E\delta_{m,n} \tag{33a}$$

$$\sum_m f^k_{Xm}(x) f^l_{Xm}(y) = P_X^{kl}\delta(x, y) \tag{33b}$$

where P_X^{kl} is a projector to the appropriate part of the tangent space of H. The parameter E is equal to ± 1 depending on the nature of the vector field in question. It is important to notice that the orthonormality conditions hold only at the surface X^3.

The expression for the metrix tensor in these coordinates can be deduced by expanding the infinitesimal deformations DH^K in the basis $\{f^K_{Xm}\}$

$$DH^K = \sum_{X,m} f^K_{Xm}\, dc^m_X \tag{34a}$$

$$dc^m_X = \int_X h_{kl} f^k_{Xm} DH^k\, dV \tag{34b}$$

The line element can be written thus written in the form

$$DS^2 = \sum_X H^X_{mn}\, dc^m_X\, dc^n_X \tag{35a}$$

$$H^X_{mn} = H_{KL} f^K_{Xm} f^L_{Xn} \tag{35b}$$

where the tensors H^X_{KL} can be identified as various parts of the proper metric tensor and summation over indices implies integration over the coordinate variables.

The orthonormalization conditions imply that the metric diagonalizes at the surface X^3 and that its elements are equal to ± 1 depending on wether the particular vector field is time- or spacelike.

The functional derivatives of the quantities f^K_{Xm} vanish and thus the transition to the coordinate representation is analogous to a linear coordinate change so that also metric connection and thus all connections transform as tensor objects in the transition.

The volume element of $SH(1)$ is given by the expression

$$DV = \prod_X [\mathrm{DET}\,(H_X)]^{1/2} \prod_m dc^m_X \tag{36}$$

in the neighborhood of X^3. Since the metric equals to unit matrix at the point X^3 of $SH(t, 1)$ the condition $\mathrm{DGT}\,(H) = 1$ holds at X^3.

4. SUPER FIELD FORMULATION

The generalization of the concept of complex-valued probability amplitude to a Grassmann algebra valued probability amplitude ("scalar super field") offers an attractive possibility to describe both bosonic and fermionic states with arbitrary high fermion number using a single fieldlike quantity.

In this section we shall develop the concept of the scalar superfield in finite-dimensional case. We treat the finite-dimensional case first, not only because the treatment of the infinite-dimensional case reduces to a mere generalization of the finite-dimensional formalism but also because the concept of the scalar superfield to be introduced is not identical with that one commonly used in the context of the supersymmetric field theories (Volkov and Akulov, 1973; Wess and Zumino, 1974; Stelle, 1983). The generalization of the formalism to the infinite-dimensional case is performed and the uniqueness of the superfield d'Alambertian is discussed.

4.1. Finite-Dimensional Case

The key concept of the finite-dimensional super scalar field formulation is the local Grassmann algebra (Berezin, 1966) spanned by the "theta parameters," which are in 1:1 correspondence with the spinor basis associated with a given point of H; the superfield is simply a map associating with each point of H an element in this algebra. In the formulation of the superfield dynamics the concept of "supercovariant derivative" changing fermion number by one unit and that of "super d'Alambertian" as a generalization of the ordinary d'Alambertian are in a key role.

4.1.1. Definition of the Scalar Superfield

Consider first the definition of the Grassmann algebra structure. Let $\{u_\alpha\}$ be a complete orthonormalized basis of spinors at the point h of H. Define the conjugate basis $\{\bar{u}_\alpha\}$ via the formula

$$\bar{u}_\alpha = (u^* \gamma^0)_\alpha \tag{37a}$$

where γ^0 is a flat space gamma matrix with the property that the condition

$$\gamma^0 \gamma^A \gamma^0 = (\gamma^A)^+ \tag{37b}$$

holds.

Associate with each element $u_\alpha(\bar{u}_\alpha)$ an anticommuting theta parameter $\theta_\alpha(\bar{\theta}_\alpha)$. The requirement that the quantitices $\bar{\theta}_\alpha u^\alpha$ and $\theta_\alpha \bar{u}^\alpha$ are invariant under vielbein rotations implies that theta parameters transform as spinors under vielbein rotations.

Theta parameters generate a Grassmann algebra at the point h. By "globalizing" this concept one is led to the concept of the "Spinorial Grassmann algebra bundle" having as its fiber the local Grassmann algebra generated by the theta parameters. The spinorial Grassmann algebra bundle might be regarded as a spin-1/2 version of the Grassmann algebra bundle generated by 1-forms of H.

Any element of the Grassmann algebra can be expressed as a polynomial of the theta parameters:

$$S = \sum_{M,N} S_{\alpha_1 \cdots \alpha_M \beta_1 \cdots \beta_N} \theta^{\alpha_1} \cdots \theta^{\alpha_M} \bar{\theta}^{\beta_1} \cdots \bar{\theta}^{\beta_N} \tag{38}$$

The coefficients of the various monomials are complex numbers and behave as multispinors under vielbein rotations since the Grassmann algebra elements must be invariant under vielbein rotations. In the sequel we shall use the following shorthand notation for the expansion of the scalar superfield.

$$S = S_{M,M}\theta^M\bar{\theta}^N \tag{39}$$

For the sake of completeness we restate the standard formulas defining integration in Grassmann algebra (Berezin, 1966):

$$\int \theta_\alpha \, d\theta_\beta = \delta_{\alpha,\beta}, \text{ etc.} \tag{40a}$$

$$\int d\theta_\alpha = \int d\theta_\alpha \, \bar{\theta}_\beta = 0, \text{ etc.} \tag{40b}$$

The integration measure in Grassmann algebra is given by

$$D\theta D\bar{\theta} = \prod_\alpha d\theta_\alpha \, d\bar{\theta}_\alpha \tag{41}$$

The scalar superfield can be defined as a map which associates to each point of the space H an element of the Grassmann algebra associated with that point. In a more advanced formulation the superfield is a section in the Grassmann algebra bundle. The component representation of the scalar superfield is obtained from the polynomial representation of the Grassmann algebra element.

An important feature differentiating between the concept of the scalar superfield used in the supersymmetric field theories and in the present context is that now the components of the scalar superfield are assumed to be complex numbers; in supersymmetric field theories the odd components are assumed to be anticommuting numbers (Volkov and Akulov, 1973; Wess and Zumino, 1974; Stelle, 1983; Berezin, 1966).

The conjugation operation of the spinors can be generalized in an obvious way to a conjugation of the scalar superfield. The conjugation is performed according to the following rules:

(i) Contract each index of each superfield component with the matrix γ^0.

(ii) Perform complex conjugation.

(iii) Replace θ_α with $\bar{\theta}_\alpha$ and vice versa.

We shall denote the conjugate of the scalar superfield S with the symbol $\bar{S}$.

A natural "scalar product" for two superfields S_1 and S_2 is given by the formula

$$(S_1, S_2) = \int \bar{S}_1 S_2 D\theta D\bar{\theta}\sqrt{h} \, d^h h \tag{42}$$

5.1.2. Definition of Supercovariant Derivatives

We define the supercovariant derivatives as direct generalizations of those introduced in the context of the Minkowski space supersymmetric field theories (Volkov and Akulov, 1973; Wess and Zumino, 1974; Stelle, 1983; Berezin, 1966)

$$D^{\varepsilon}_{\alpha} = \frac{\partial}{\partial \theta^{\alpha}} + \varepsilon i \bar{B}^{k}_{\alpha} D_k \tag{43a}$$

$$\bar{D}^{\varepsilon}_{\alpha} = -\frac{\partial}{\partial \bar{\theta}_{\alpha}} - \varepsilon i B^{k}_{\alpha} D_k \qquad (\varepsilon = \pm 1) \tag{43b}$$

$$B^{k}_{\alpha} = (\Gamma^{k}\theta)_{\alpha} \tag{43c}$$

$$\bar{B}^{k}_{\alpha} = (\bar{\theta}\Gamma^{k})_{\alpha} \tag{43d}$$

The derivative D_k is the usual covariant derivative containing spinor connection part and metric connection part. Theta parameters are by definition covariantly constant with respect to the covariant derivative D_k and the components of the scalar superfield transform as multispinors, that is, like tensor products of the appropriate number of ordinary spinors.

The supercovariant derivatives defined act from the left. In the sequel also the supercovariant derivatives acting from the right will be needed and we shall use the symbols $\vec{D}^{\varepsilon}_{\alpha} \equiv D^{\varepsilon}_{\alpha}$ and $\overleftarrow{D}^{\varepsilon}_{\alpha}$ for the left and right derivatives, respectively.

The following remarks concerning the properties of the supercovariant derivatives should be made.

(i) The derivatives $\vec{D}^{\varepsilon}_{\alpha}$ and $\overleftarrow{D}^{\varepsilon}_{\alpha}$ are Hermitian conjugates of each other with respect to the above defined scalar product in the space of the superfields.

(ii) The order of the various factors in the terms B^{k}_{α} and $\bar{B}^{k}_{\alpha}$ is unessential, since gamma matrices and theta parameters are covariantly constant. It should be noticed, that in the case of $H = M^4 \times CP_2$ the covariant derivative must contain also the Kähler potential.

(iii) The anticommutators of the supercovariant (left) derivatives are given by the formulas

$$\{D^{\varepsilon}_{\alpha}, D^{\delta}_{\beta}\} = -\varepsilon\delta \bar{B}^{k}_{\alpha} \bar{B}^{l}_{\beta} F_{kl} \tag{44a}$$

$$\{\bar{D}^{\varepsilon}_{\alpha}, \bar{D}^{\delta}_{\beta}\} = -\varepsilon\delta B^{k}_{\alpha} B^{l}_{\beta} F_{kl} \tag{44b}$$

$$\{D^{\varepsilon}_{\alpha}, \bar{D}^{\delta}_{\beta}\} = -i(\varepsilon + \delta)\Gamma^{k}_{\alpha\beta} D_k + \varepsilon\delta B^{k}_{\alpha} B^{l}_{\beta} F_{kl} \tag{44c}$$

$$F_{kl} = \{D_k, D_l\} \tag{44d}$$

When space H is flat so that the curvature form of the spinor connection vanishes, the spinor covariant derivatives D^{ε}_{α} and D^{ε}_{β} anticommute and D^{ε}_{α} and $\bar{D}^{\varepsilon}_{\beta}$ anticommute to a translation.

4.1.3. Super d'Alambertians

The most general super d'Alambertian in the case of nonflat space is expected to be of the following form

$$\boxtimes = \sum_{\varepsilon} a_{\varepsilon} \boxtimes_{\varepsilon} + b_{\varepsilon} \bar{\boxtimes}_{\varepsilon} \tag{45a}$$

$$\boxtimes_{\varepsilon} = D^{\varepsilon}_{\alpha} \bar{D}^{\varepsilon}_{\alpha} \tag{45b}$$

$$\bar{\boxtimes}_{\varepsilon} = \bar{D}^{\varepsilon}_{\alpha} D^{\varepsilon}_{\alpha} \tag{45c}$$

where the parameters a_{ε} and b_{ε} are real numbers.

When the space H is flat the operators $\boxtimes_{\varepsilon}$ and $-\bar{\boxtimes}_{\varepsilon}$ are equal as the equation

$$\boxtimes_{\varepsilon} + \bar{\boxtimes}_{\varepsilon} = \bar{\theta} \Sigma^{kl} F_{kl} \theta \tag{46}$$

The operator $\boxtimes$ is indeed derivable from a variational principle defined by a Lagrangian, which is (i) Hermitian with respect to the conjugation $S \to \bar{S}$ and (ii) quadratic with respect to the superfield.

Since the covariant derivatives $\vec{D}^{\varepsilon}_{\alpha}$ and $\overleftarrow{\bar{D}}^{\varepsilon}_{\alpha}$ are Hermitian conjugates with respect to the scalar product defined earlier [formula (42)] the following variational principle satisfies these requirements and leads to the expected form of the super d'Alambertian:

$$S = \int L D\theta D\bar{\theta} \sqrt{h}\, d^{n}h \tag{47}$$

$$L = \sum_{\varepsilon} \bar{S}(a_{\varepsilon} \overleftarrow{D}^{\varepsilon}_{\alpha} \vec{\bar{D}}_{\alpha} + b_{\varepsilon} \overleftarrow{\bar{D}}^{\varepsilon}_{\alpha} \vec{D}^{\varepsilon}_{\alpha}) S \tag{48}$$

The various arrows appearing in the formulas give the direction of the action of the supercovariant derivative in question.

Notice that one can either (i) regard the field formed by the components of the superfield as a set of fields in H and satisfying the field equations derivable from the Lagrangian obtained by integrating over the theta parameters in the super Lagrangian (so that the introduction of the anticommuting theta parameters can be regarded as an effective book keeping trick), or (ii) regard also the theta parameters as arguments of the superfield and derive the equations of motion using the generalization of the ordinary rules to obtain field equations from a given Lagrangian density.

The field equations are expectedly given by

$$\boxtimes S = 0 \tag{49}$$

A nice way to get rid of the nonuniqueness related to the choice of the super d'Alambertian is based on the observation that the super d'Alambertians $\boxtimes_{\varepsilon}$ allow a large set of vacuum solutions since any superfield annihilated by the rightmost supercovariant derivative, and thus satisfying one of the conditions

$$D^{\varepsilon}_{\alpha}S=0 \tag{50a}$$

$$\bar{D}^{\varepsilon}_{\alpha}S=0 \tag{50b}$$

is a vacuum solution of the equations of motions as the study of a general expression for the conserved quantities reveals.

The solutions satisfying this condition (known as chiral condition in literature; Volkov and Akulov, 1973; Wess and Zumino, 1974; Stelle, 1983) generate symmetries analogous to Abelian gauge transformations since one can add to an arbitrary solution of super d'Alambertian a chiral superfield and thus eliminate unphysical degrees of freedom. Hence we shall call this symmetry Abelian super gauge invariance.

Since the requirement of maximal symmetry fixes the super d'Alambertian essentially uniquely (the four maximally symmetric operators are physically equivalent), we shall in the sequel restrict our considerations to the super d'Alambertians of this type.

4.1.4. Symmetries of the Superaction

In the discussion of the symmetries we assume for definiteness that super d'Alambertian is of the form $\boxtimes_{\varepsilon}$, $\varepsilon=1$ and shall drop the indices ε in the sequel from the various formulas.

When space H is flat the general solution to the chiral condition

$$D_{\alpha}S=0 \tag{51}$$

is given by the expression

$$S=S_1S_2 \tag{52a}$$

Here S_1 is an arbitrary function of the argument

$$X^k=h^k+iY^k \tag{52b}$$

$$Y^k=\bar{\theta}\Gamma^k\theta \tag{52c}$$

and the superfield S_2 is of the form

$$S_2=S_M\theta^M \tag{52d}$$

Here the coefficients are constant multispinors.

From the anticommutativity of the derivatives D_{α} and $\bar{D}_{\alpha}$ (when the space H is flat) it follows that also the fields satisfying the condition

$$\bar{D}_{\alpha}S=0 \tag{53}$$

are vacuum solutions. The general solution to the condition (53) is obtained by the replacements

$$X^k \to h^k - iY^k$$

$$\theta_\alpha \to \bar{\theta}_\alpha$$

in the formulas (52a) and (52b), respectively.

The solution of the chiral condition (51) in nonflat case is of the same general form given by the equation (52a).

The field S_1 is obtained by the following recipe:

(i) Taken an arbitrary function S_1 of the argument X^k defined by the equation (52b) and thus satisfying the chirality condition in flat case.

(ii) Expand this function in Taylor series with respect to the argument Y^k defined in (52c)

(iii) Replace the ordinary derivatives appearing in the coefficients of the expansion with covariant derivatives.

The superfield S_2 is of the same form as in the flat case, the multispinor being now covariantly constant (and representable as a sum over products of covariantly constant spinors).

It is important to notice that the covariant derivative $\bar{D}_\alpha$ does not anticommute with D_α and therefore super gauge invariance is lost if a term proportional to $\boxtimes_\varepsilon$ is added to the super d'Alambertian unless the space H is flat.

When the space H is flat the super d'Alambertian $\boxtimes_\varepsilon$ allows also supersymmetries as its symmetries (Volkov and Akulov, 1973; Wess and Zumino, 1974; Stelle, 1983). The supersymmetry transformations are defined as transformations of the form

$$S \to S\exp(tJ) \qquad (54a)$$

$$J = \bar{\psi}_{1\alpha}\overleftarrow{D}_\alpha + \psi_{2\alpha}\overleftarrow{\bar{D}}_\alpha \qquad (54b)$$

The quantities $\bar{\psi}_1$ and ψ_2 are constant complex spinors rather than anticommuting Grassmann numbers unlike the corresponding parameters appearing in the supersymmetries of the supersymmetric field theories. The operator J can be regarded as the infinitesimal generator of the symmetry and the arrows imply that the action of the supercovariant derivative is from right (parameter t is real).

The nicest way to see that these transformations are symmetries is to show that the infinitesimal generator of the supersymmetry commutes with the supercovariant derivatives D_α and $\bar{D}_\alpha$

$$[D_\alpha, J] = 0 \qquad (55a)$$

$$[\bar{D}_\alpha, J] = 0 \qquad (55b)$$

These conditions are indeed satisfied since the spinors appearing in the transformation formula are constant and since the various left and right supercovariant derivatives appearing in the commutator commute, when the space H is flat. The general expressions for the various commutators are given by

$$[D^{\varepsilon}_{\alpha}, \overleftarrow{D}^{\delta}_{\beta}] = \varepsilon\delta\bar{B}^{k}_{\alpha}F_{kl}\bar{B}^{l}_{\beta} \tag{56a}$$

$$[D^{\varepsilon}_{\alpha}, \overleftarrow{\bar{D}}^{\delta}_{\beta}] = -2iE(\varepsilon+\delta)\Gamma^{k}_{\alpha\beta}D_{k} + E\varepsilon\delta\bar{B}^{k}_{\alpha}F_{kl}B^{l}_{\beta} \tag{56b}$$

as one verifies by a direct calculation. The factor E is a sign factor which equals to $+1$ and -1 for even and odd components of the superfield, respectively. The quantities B^{k}_{α}, B^{l}_{β}, and F_{kl} are defined in the formulas (43) and (44).

The transformation obtained by replacing the spinors appearing in the flat space transformation formula with covariantly constant ones is an obvious candidate for the supersymmetry transformation in the nonflat case.

In the general case space H does not allow covariantly constant spinors. Furthermore, from the formulas (56) it is also clear that the curvature of the space H breaks supersymmetry even in the case when covariantly constant spinors exist. Thus supersymmetry resembles Poincarè invariance in the sense that the curvature of the space H breaks this symmetry.

It should be emphasized that the concept of the supersymmetry in the present context differs from that used in the context of the supersymmetric field theories. First, the parameters appearing in the supersymmetry transformation are not Grassmann numbers. Second, supersymmetries are symmetries in the conventional sense of the word. The differences derive from the fact that the coefficients appearing in the expansion of the TGD superfield are ordinary complex numbers.

4.2. Generalization to Infinite-Dimensional Case

The generalization of the finite-dimensional formulation looks quite straightforward; one needs only to replace various finite index sets with infinite ones and, if needed, replace summations with integrations. There are however some complications resulting from the delicacies associated with the spinor structure of $M^4 \times CP_2$.

4.2.1. Definition of the Grassmann Algebra Structure

The definition of the Grassmann algebra structure in SH is a quite straightforward generalization of the finite-dimensional definition.

Consider first the problem of defining a conjugation operation for the spinors of $SH(1)$. Denote by the symbol X Int X^3 or any of its boundary components and let $\{\psi^{m}_{X}\}$ be the spinor basis associated with X.

Since the spinors of $SH(1)$ are maps, which associate an H-spinor field to each X, it is natural to define the action of the spinor conjugation as that induced by the H-spinor conjugation defined earlier by the formulas (37).

Thus the action of the spinor conjugation of the X part of the $SH(1)$ spinor is defined by the formulas

$$\bar{\psi}_X^m = (\psi_X^* \Gamma_X^0)^m \tag{57a}$$

where the matrix Γ_X^0 is defined by the equation

$$\Gamma^0_{Xm_n} = (\psi^m_{X_1} \gamma^0 \psi^n_X) \tag{57b}$$

Observe that also the action of the discrete "symmetries" (Pitkänen, 1981, 1983) on $SH(1)$ spinors can be defined by inducing their known action on spinors of H.

Associate with each spinor field basis $\{\psi_X^m\}$ the sets $\{\theta_X^m\}$ and $\{\bar{\theta}_X^m\}$ of theta parameters transforming under vielbein rotations as spinors of $SH(1)$ and generating an infinite-dimensional Grassmann algebra, denoted by the symbol GR in the sequel.

A general element of the Grassmann algebra generated by the theta parameters is an infinite power series with respect to the various theta parameters. The coefficients of the series are complex-valued functionals of the surface X^3 transforming as multispinors in $SH(1)$.

The definition of the Grassmann algebra conjugation is identical with the finite-dimensional definition. The Grassmann algebra integration measure is defined to be the product of the integration measures associated with the various boundary components and interior:

$$D\theta D\bar{\theta} = \prod_X D\theta_X D\bar{\theta}_X \tag{58a}$$

$$D\theta_X D\bar{\theta}_X = \prod_m d\bar{\theta}_X^m \, d\theta_X^m \tag{58b}$$

Define superfield as a map from $SH(1)$ to the Grassmann algebra GR. The component representation of the superfield is given by

$$S = S_{M'N} \bar{\theta}^M \theta^N \tag{59}$$

In a more advanced formulation one should obviously define the superfield as a section in the "bundle" having as a fibre the Grassmann algebra generated by the theta parameters associated with interior and various boundary components. Notice however that the fiber spaces corresponding to different manifold topologies need not be isomorphic so that the structure in question is probably more general than bundle structure.

4.2.2. *Definition of the Super d'Alambertian*

The generalization of the formulas defining the supercovariant derivatives is straightforward. The supercovariant derivatives are defined by the formula

$$D_m = \frac{\partial}{\partial \theta_m} + i\bar{B}_{mK} H^{KL} D_L \tag{60a}$$

$$\bar{D}_m = \frac{\partial}{\partial \bar{\theta}_m} - iB_{mK} H^{KL} D_L \tag{60b}$$

$$\bar{B}_{mK} = (\bar{\theta}\Gamma_K)_m \tag{6c}$$

$$B_{mK} = (\Gamma_K \theta)_m \tag{60d}$$

The formula for the spinor connection contains interior part and various boundary parts; in particular, it contains terms, which couple the interior and boundary degrees of freedom to each other.

The action of the functional derivatives on the state functionals (in particular to the parts of the state functionals depending only on a single boundary component) can be derived by considerations similar to those used to derive expressions for metric and vielbein connections.

It is important to notice that no explicit coordination of the space $SH(1)$ has been introduced to define the supercovariant derivatives.

The generalization of the super d'Alambertian operators is straightforward, when the various theta parameters are regarded as independent variables. The expressions for the general super action and for the general super d'Alambertian are formally identical with the finite-dimensional expressions once the index identification $K=(k, x)$ is made and index summation is replaced with an integral over x and summation over k.

The requirement of the maximal super gauge invariance leads to a unique theory in general case. In case of $M^4 \times CP_2$ the addition of a CP-breaking term into the action is however possible; we shall later show that the requirement of maximal super gauge invariance fixes this term uniquely.

4.2.3. *Extension of the Results to* $SH(n)$

Hitherto we have defined the various concepts only for the case of $SH(1)$. The extension of the results obtained to a general case is, however, straightforward since $SH(n)$ is isomorphic to the Cartesian product of n $SH(1)$:*s* divided by the symmetric group S_n.

The Grassmann algebra structure for $SH(n)$ can be defined as the tensor product of the Grassmann algebras associated with different factors.

The total Hilbert space of spinors is spanned by the direct sum of the spinor spaces associated with different 3-manifolds. The theta parameters associated with various summands of course are assumed to anticommute.

The superfields are generated by the products of the superfields associated with the different factors and the super d'Alambertian for $SH(n)$ separates into a sum of the super d'Alambertians associated with the different factors.

$$\boxtimes = \sum_i \boxtimes_i \tag{61}$$

Thus the single particle state functionals must be eigen states of the respective super d'Alambertians and the sum of the eigenvalues must vanish:

$$\boxtimes_i S_i = l_i S_i \tag{62a}$$

$$\sum_i l_i = 0 \tag{62b}$$

It is clear that the only physically allowed eigenvalues are vanishing since the mere presence of the other particles cannot change the spectrum of one-particle states. The eigenvalues for the various super d'Alambertians need not vanish unless one poses some special condition to the superspinors.

It is easy to invent this kind of condition. One can pose chirality condition on spinors of H:

$$\Gamma_9 \psi = \varepsilon \psi, \qquad \varepsilon = +1/-1 \tag{63}$$

In the theory without chirality condition both leptons and quarks are assumed to be elementary fermions and they correspond to two different chiralities possible for H spinors.

In the simplest scenario satisfying the chirality condition leptons are the only elementary fermions, and quarks correspond to leptons in "pseudotriplet" color partial waves as found in the first part of the series.

The vanishing of the eigenvalues associated with the various single-particle super d'Alambertians is guaranteed by posing and chirality condition on the superfield. This means that we define the matrix elements of the matrix Γ^l_X via the formula

$$(\Gamma^9_X)_{mn} = (\psi^m_{X1} \Gamma_9 \psi^n_X) \tag{64}$$

and require that the contraction of Γ^9_X with any index of the superfield is $+1$ or -1 depending on whether the index is fermionic or antifermionic.

This condition implies that super d'Alambertian changes the chirality of the each component of the superfield and thus the only possible eigenvalue of the super d'Alambertian is equal to zero.

Thus a rather unexpected and physically highly nontrivial result follows from the mere requirement that that the theory has multiplicative superposition in the sense that the solutions to the equations of motion in $SH(n)$ are superpositions of the products of single particle solutions to the equations of motion.

A second nice feature of the chirality condition is that field equations simplify considerably since the operator $\partial\theta_m\partial\bar{\theta}_m$ annihilates the superfield identically. The reason is that this operator creates contractions between fermionic and antifermionic indices and these contractions vanish when the chirality condition is satisfied.

4.2.4. Extension of the Results to the Singular Manifolds

We have now generalized the superfield formalism so that it applies in subsets of SH consisting of nonsingular 3-manifolds. Concerning the description of the interactions the subsets $SH(t1, t2)$ consisting of three-surfaces having topologies intermediate between manifold topologies $t1$ and $t2$ play a central role.

Since the superfields associated with the topologies $t1$ and $t2$ must be comparable in the singular limit there must be a relationship between the theta parameters associated with these topologies.

This relationship can be derived by relating to each other the spinor bases associated with the topologies $t1$ and $t2$ at the singular limit and equating the quantities $\bar{\psi}^m_X\theta_m$ and $\psi^m_X\bar{\theta}_m$ associated with $t1$ and $t2$ in this limit.

Let us derive the relationship between the theta parameters in some special cases.

(a) The decay of a closed 3-manifold to two closed 3-manifolds. Denoting the decaying 3-manifold by $s1$ and the product manifolds by $s2$ and $s3$ we have the following relationship between the various spinor bases:

$$\psi^1_m = C^{2n}_m\psi^2_n + C^{3n}_m\psi^3_n \tag{65}$$

The matrix relating the conjugate spinors is obtained from the matrix C^{kn}_m by complex conjugation.

The coefficients C^{kn}_m are given by the overlap integrals

$$C^{kn}_m = \int_{s_k} (\psi^k_n)^*\psi^1_m\, d^dx \qquad (k=2,3) \tag{66}$$

The relationship between the theta parameters is determined by requiring the continuity of the 1-fermion superfields

$$\psi^m_1\bar{\theta}^1_m = \psi^m_2\bar{\theta}^2_m + \psi^m_3\bar{\theta}^3_m \tag{67}$$

This condition holds provided the "barred" theta parameters are related via the following formula:

$$\bar{\theta}^k_m = C^{kn}_m \bar{\theta}^1_n \qquad (k=2,3) \tag{68}$$

The matrix relating theta parameters is a complex conjugate of the matrix $\bar{C}^{kn}_m$.

(b) The decay of a single boundary component to two boundary components. It is clear that the theta parameters associated with the initial boundary component and the final boundary components are related in a similar manner as in the higher-dimensional case already treated. The theta parameters associated with the interiors are deduced by requiring the continuity of the 1-fermion parts of the superfields.

The relationship between the interior theta parameters is thus given by the formula

$$\bar{\theta}^2_m = C^n_m \bar{\theta}^1_n \tag{69}$$

where the coefficients C^n_m are given by the overlap integrals

$$C^n_m = \int_{X^3} (\psi^2_n)^* \psi^1_m \, d^3x \tag{70}$$

5. *CP*-BREAKING MODIFICATION OF THE SUPER D'ALAMBERTIAN

5.1. General Considerations

One of the most exciting features of the proposed approach is its predictivity; the requirement of the maximal super gauge invariance leads to unique super d'Alambertian in the general case. In the case of the space $M^4 \times CP_2$ one can however imagine a modification of the super d'Alambertian made possible by the special geometric properties of CP_2.

As shown in the first paper of the series, the covariantly constant Kähler form of CP_2 (Eguchi et al., 1980; Gibbons and Pope, 1978; Hawking and Pope, 1978) allows a modification of the ordinary Dirac equation for the spinor fields induced to a surface of H since one can replace the gamma matrices Γ_k or CP_2 with modified gamma matrices

$$\tilde{\Gamma}_k = (h^l_k + ilJ^l_k)\Gamma_l \tag{71}$$

Here l is an arbitrary real number.

It is clear that this replacement allows to define a modified Dirac operator in CP_2 and H; also one can obtain a modification of the super d'Alambertian in H and in SH via this replacement.

What makes this modification so interesting is that it is CP-breaking (metric and Kähler form are CP-even and CP-odd, respectively). Thus the special geometric properties of CP_2 might provide an explanation for the mysterious CP-breaking effects observed in Nature (Cronin, 1981; Fitch, 1981).

In the sequel we shall first define the modified gamma matrices, Dirac operator, and super d'Alambertian in CP_2, H, and SH. Furthermore, we shall show that for two special values of the parameter l $(=\pm 1)$ (i) it is possible to define a cohomology theory for the spinor fields of CP_2 (and also of $M^4 \times CP_2$), and (ii) the Abelian super gauge invariance associated with the modified super d'Alambertian $\tilde{D}_\alpha \bar{\tilde{D}}_\alpha$ is exceptionally large since the chiral superfields $(\tilde{D}_\alpha \Omega = 0)$ have (in a certain sense local) supersymmetries as their dynamical symmetry group.

5.2. Modified Gamma Matrices, Dirac Operators, Etc.

In order to define the modified gamma matrices and to understand their basic properties recall that the so-called Kähler form J defines in CP_2 symplectic structure analogous to the symplectic structure of the phase space of classical mechanics.

The two defining features of the Kähler form needed in the sequel are the following ones:

(i) The Kähler form is covariantly constant.

(ii) The square of the Kähler form is the negative of the metric tensor

$$J_{kr} J^r_l = -s_{kl} \tag{72}$$

Since Kähler form is covariantly constant the modified gamma matrices defined by the equation (71) are covariantly constant in CP_2 (H).

The anticommutator of the modified gamma matrices of H is given by the formula

$$\{\tilde{\Gamma}_k, \tilde{\Gamma}_l\} = 2(m_{kl} + (1 - l^2)s_{kl}) \tag{73}$$

This representation follows from the equation (72).

For the special choice $l = 1/-1$ these matrices commute to Minkowski metric noninvertible as a matrix in H.

One can extend this definition also to the case of $SH(1)$. The modified vielbein and gamma matrices of $SH(1)$ are obtained by the replacement

$$e^A_k \to \hat{e}^A_k \tag{74}$$

in the general formulas defining the vielbein and gamma matrices of $SH(t, 1)$.

Here the modified vielbein $\tilde{e}_k^A$ of H is defined via the formula

$$\tilde{e}_k^A = (h^l_{\ k} + ilJ^l_{\ k})e_l^A \tag{75}$$

The anticommutators of the modified gamma matrices are given by

$$\{\tilde{\Gamma}_K, \tilde{\Gamma}_L\} = 2\tilde{H}_{KL} \tag{76a}$$

$$\tilde{H}_{KL} = \tilde{H}_{kl}(x, y) = (h_{sr} - l^2 s_{rs})G^{rs}_{kl} \tag{76b}$$

$$G^{rs}_{kl} = \delta^3(x, y)P^r_k P^s_l + \sum_{\delta} \delta^3(x, f(\xi))\delta^3(y, f(\xi))P^r_{\delta k}P^s_{\delta l}\, d^2\xi \tag{76c}$$

Again it is found that for the special choices of the parameters the communtator is proportional to the appropriate projection of the Minkowski metric.

The definition of the modified Dirac operator is obvious; one needs only to replace the gamma matrices appearing in these operators with the modified gamma matrices. It should be noticed that for $l = +1/-1$ the square of the modified Dirac operator of CP_2 vanishes identically. As a consequence this operator defines "spinor cohomology" in $M^4 \times CP_2$ to be studied in more detail in the sequel.

It should be noticed that the modified Dirac operator is not derivable from an action principle. The requirement of Hermiticity implies the form

$$L = \bar{\psi}(\tilde{\Gamma}(l)\vec{D}_k - \overleftarrow{D}_k\tilde{\Gamma}^k(-l))\psi \tag{77}$$

for the Lagrangian density differing from the ordinary Dirac Lagrangian density only by a total divergence.

For spinors induced on a submanifold of H the Lagrangian of the form given by (77) is however not equivalent with the ordinary Dirac Lagrangian as found in the first paper of the series.

It is important to notice that $D^+_\alpha(l)$ and $\bar{\overleftarrow{D}}(l)$ are not Hermitian conjugates of each other since Hermitian conjugation effectively changes the sign of the parameter l:

$$(D^+_\alpha(l)) = \bar{\overleftarrow{D}}(-l) \tag{78}$$

Thus the supercovariant derivatives must appear in combinations of type $D(l)\bar{D}_\alpha(-l)$, etc.... in the CP-breaking super d'Alambertian.

As already mentioned the modification of the gamma matrices makes makes theory CP asymmetric. Since CP operation corresponds to complex conjugation in CP_2 and since the metric and Kähler form of CP_2 are CP-even and -odd, respectively, the CP operation effectively changes the sign of the parameter l.

5.3. Spinor Cohomology

As already noticed the values $+1$ and -1 of the parameter l are in a distinguished position mathematically since the gamma matrices of H anticommute to Minkowski metric for this special values of l and the square of the modified Dirac operator of CP_2 vanishes. This operator, denoted by the symbol d in the sequel, thus defines "spinor cohomology" in H. One can define closed ($d\psi = 0$), exact ($\psi = d\psi_1$) and cohomologically nontrivial (closed but nonexact) spinors. Furthermore, one can define cohomology group as the linear space of the cohomologically nontrivial spinors.

In order to understand the properties of the spinor cohomology it is advantageous to use complex coordinates (ξ^1, ξ^2, $\bar{\xi}^1$, $\bar{\xi}^2$) for CP_2 (see Pitkänen, 1983, and Eguchi et al., 1980). In these coordinates the operator d has a surprisingly simply form:

$$d = 2\Gamma^{\bar{k}} D_{\bar{k}} \tag{79}$$

Thus d is simply proportional to that half of the Dirac operator, which acts on the variables $\bar{\xi}^k$.

The square of d is given by the formula

$$d^2 = 4\Sigma^{\bar{k}\bar{l}} F_{\bar{k}\bar{l}} \tag{80}$$

and vanishes since the curvature form of the spinor connection satisfies the conditions

$$F_{kl} = F_{\bar{k}\bar{l}} = 0 \tag{81}$$

as can be verified by a direct calculation using the expression for the curvature form of the spinor connection given in Pitkänen (1983) and Eguchi et al. (1980).

From the representation of d in complex coordinates one can derive important information concerning the properties of closed spinors.

(i) The set of closed spinors is closed with respect to the multiplication with analytic functions (functions of the variables ξ^k only).

(ii) The spinors satisfying the condition

$$D_{\bar{k}}\psi = 0 \tag{82}$$

are closed and good candidates for cohomologically trivial spinors.

Clearly, these conditions can be regarded as a generalization of the analyticity conditions obtained by replacing ordinary derivatives with covariant derivatives.

The integrability conditions associated with these equations are satisfied identically (!) by equation (80).

Covariant constancy conditions can be solved in closed form. Equations (81) imply that spinor connection can be written in the form

$$A_{\bar{k}} = g g^{-1}_{|\bar{k}} \tag{83a}$$

$$A_k = h h^{-1}_{|k} \tag{83b}$$

where g and h are elements of the gauge group of the spinor connection.

The covariant constancy conditions can be written in the form

$$(\partial_{\bar{K}} + g g^{-1}_{|\bar{K}})\psi = 0 \tag{84}$$

The spinors satisfying these conditions are of the form

$$\psi = g\psi_1(\xi^k) \tag{85}$$

Thus these spinors are apart from the "gauge factor" g analytic functions of the variables ξ^k and we shall call them analytic spinors.

We believe that the analytic spinors are just the cohomologously non-trivial spinors in spinor cohomology. It is probably easy to verify the correctness (or incorrectness) of this belief by explicit comparison of the exact spinors to analytic spinors.

5.4. Extended Super Gauge Invariance of the Superaction in Critical Case

In critical case ($l = 1$ for definiteness) the modified super d'Alambertian $\boxtimes = D_\alpha(-1)\bar{D}_\alpha(1)$ (and any of the eight physically equivalent super d'Alambertians of this type) is characterized by an exceptionally large super gauge invariance.

The extended Abelian super gauge invariance is closely related to the commutativity of the operator $\bar{D}_\alpha(1)$ (but not the super d'Alambertian!) appearing in the chiral condition with super symmetries generated by analytic spinors satisfying the condition

$$D_{\bar{K}}\psi = 0 \tag{86}$$

The general form of the supersymmetry is given by

$$S \to S \exp(\mathrm{tJ}) \tag{87a}$$

$$J = \psi_1^\alpha A_\alpha + \psi_2^\alpha B_\alpha \tag{87b}$$

$$A_\alpha = \vec{\bar{D}}_\alpha(1), \qquad B_\alpha = \overleftarrow{\bar{D}}_\alpha(1) \tag{87c}$$

Here ψ_1 and ψ_2 are analytic spinors and the operator $\exp(tJ)$ acts from the right.

The condition for the supersymmetry invariance of the chiral condition is the commutativity of the operators A_α and B_α with the supercovariant derivative $\bar{D}_\alpha(1)$.

The commutativity follows from the absence of the derivatives D_k in the operators A_α and B_α (complex coordinates are used!). First, the analytic spinors are effectively covariantly constant. Furthermore, the commutators $[\bar{D}_\alpha, A_\beta]$ and $[\bar{D}_\alpha, B_\beta]$ are proportional to the quantities $F_{\bar{K}\bar{L}}$, which vanish identically.

It should be noticed that super symmetries are local in the sense that one multiply the parameter t in the transformation formula by an arbitrary analytic function of the variables ξ^k.

Furthermore, supersymmetries are not symmetries of the whole super d'Alambertian, since the operator $D_\alpha(-1)$ does not annihilate analytic spinors and does not commute with A_α or B_α so that supersymmetries can be regarded as dynamical symmetries.

A very general set of chiral vacuons are generated by the superfields of type

$$S = S_1 S_2 \tag{88a}$$

Here S_1 is obtained by expanding an arbitrary function of the variables $X^k = h^k + iY^k$

$$Y^k = \theta\tilde{\Gamma}^k\theta \tag{88b}$$

in power series with respect to the variables Y^k and by replacing the ordinary partial derviatives in the coefficients of the expansion with covariant derivatives.

The superfield S_2 is of the form

$$S_2 = S_M\theta^M \tag{88c}$$

Here S_M is an arbitrary analytic multispinor having vanishing covariant derivatives with respect to the variables $\bar{\xi}^k$.

We believe that the most general chiral superfield can be constructed as a superposition of the fields of the above type although we have no proof of this conjecture.

5.5. Maximal Super Gauge Invariance Implies Unique *CP*-Breaking Super d'Alambertian

We have found that the *CP*-breaking term in the super d'Alambertian of $H = M^4 \times CP_2$ leads to a suprisingly symmetric theory for the two values of the *CP*-breaking parameter. In addition, the resulting theory is essentially unique.

The assumption that the super d'Alambertian defining the quantum theory in *SH* is of the form leading to the maximally symmetric theory in the finite-dimensional case looks very natural for several reasons:

(i) The *CP* breaking is one of the most mysterious phenomena of Nature and the idea that the requirement of maximal symmetry leads to *CP*-breaking theory is esthetically very pleasing. Of course, the phenomenon of *CP* breaking becomes a direct signature of the very special geometric properties of CP_2.

(ii) The finite-dimensional theory is expected to have something to do with the point particle limit of the real theory in SH and by maximizing the symmetries of the pointlike one maximizes also the symmetries of the real theory.

(iii) In the actual theory the supersymmetries become only approximate symmetries since there are probably no covariantly constant spinors in SH and since the concept of the spinor analyticity probably does not make sense in SH. As a consequence some degrees of freedom, which are vacuum degrees of freedom in the pointlike limit become physical degrees of freedom in the realistic theory.

This picture of symmetry breaking resembles the basic mechanisms of symmetry breaking of gauge theories based on the idea that initially gauge-like degrees of freedom physical degrees of freedom, when symmetry is spontaneously broken. An important feature differentiating between the two approaches is that in TGD symmetry breaking is caused by the curvature of the space SH and nonpointlike nature of the particles.

6. CONSTRUCTION OF S MATRIX

6.1. General Considerations

The multiplicative superposition suggests the following formal procedure for the construction of a unitary S matrix.

(1) Construction of Bare 1-Particle States. The field equations are solved in the sets $SH(t, 1)$ that in the open subset of SH consisting of connected 3-manifolds with a fixed topology t. We define the bare one-particle states as state functionals restricted to $SH(t, 1)$ and having the property that they are stationary solutions of field equations in $SH(t, 1)$.

In this context the meaning of the bare one-particle state is rather general; only the simplest 3-topologies are expected to correspond to elementary particles and the topologies obtained by forming connected sums of the simple topologies (say, by glueing particlelike 3-manifolds to a subset of a spacelike hyperplane of M^4) are expected to correspond to (gravitationally or otherwise) bound states of elementary particles.

We assume that the bare 1-particle states can be orthonormalized with respect to the conserved scalar product implied by the phase symmetry of the action and that this scalar product is positive definite, at least when restricted to the set of "physical states." Furthermore, single-particle state functionals are assumed to form a complete set with respect to this scalar product.

(2) Construction of Bare Many-Particle States. By multiplicative superposition the products of one-particle state functionals solving field equations

in $SH(t, 1)$ are solutions of the field equations in $SH(t, n)$ (set of n component 3-manifolds with fixed topologies). It is natural to define the bare n-particle states as state functionals, which vanish outside $SH(t, n)$ and are superpositions of the products of one-particle state functionals in $SH(t, n)$.

The scalar product for single-particle states defines a natural scalar product for bare many-particle states.

(3) Construction of Stationary States. Bare n-particle states are solutions of field equations both inside and outside (trivially so) $SH(t, n)$ but not in the boundaries of $\overline{SH}(t)$. Thus these states are not stationary and by the uncertainty principle are expected to disperse form $SH(t, n)$ to other parts of SH. This dispersion is observed as various particle reactions, which may change particle number and also the topological quantum numbers associated with a single particle. By multiplicative superposition the stationary states must correspond to linear superpositions of bare n-particle states in $SH(t, n)$.

There is a natural correspondence between bare and stationary states: one can construct from an n-particle bare state a stationary state by continuing this state to the other topologically different sectors of SH. The continuation of the bare n-particle state to a stationary state must be one valued. This requirement probably poses constraints on the spectrum of allowed bare states and might well lead to quantization conditions.

(4) Definition of S Matrix. Bare n-particle states resemble the incoming states of the ordinary field theories; they have sharp particle number and they are not global solutions of the field equations. The stationary states in turn resemble the outgoing states since they are stationary solutions of the field equations and have no sharp particle number. Thus it is natural to define the S-matrix as a matrix transforming the bare and stationary states to each other.

In the following sections we shall (a) formulate the continuity conditions stating that any bare state functional defined $SH(t)$ equals to superposition of bare state functionals defined in $SH(t1)$ in the set $SH(t, t1)$ that is in the set of singular manifolds common to the boundaries of $SH(t)$ and $SH(t1)$;

(b) derive a formal solution to the continuity conditions in terms of certain overlap integrals over $SH(t1, t2)$. As a result one obtains explicit expressions for couplings (say, electromagnetic coupling) as overlap integrals over $SH(t1, t2)$;

(c) derive from the one-valuedness requirement of the stationary state functionals a set of conditions, which are analogous to the conditions defining the duality concept familiar from the string models (Jacob, 1974; Chew and Rosenweig, 1978; Schwartz, 1985);

(d) derive a general expression for the S matrix.

6.2. Formal solution of the continuity conditions

In order to continue a given bare state functional defined in $SH(t1)$ to a stationary state functional in SH one can use the continuity requirement of the state functional in the sets $SH(t1, t2)$ consisting of singular manifolds intermediate between the initial and final topologies. In this section we shall formulate the continuity conditions and derive a formal solution of the conditions.

Let us first introduce some notations and definitions.

Let $V(ti)$ denote a basis of bare state functionals $S^m(ti)$ defined in $SH(ti)$, $i=1,2$. These state functionals are stationary solutions of the super d'Alambertian in the open set $SH(t)$ of SH. The phase symmetry of the super d'Alambertian implies the existence of a scalar product, which we assume to be positive definite; the super gauge invariance of the super d'Alambertian might play an important role in guaranteeing the existence of a positive definite scalar product. Thus we can assume that bare state functionals form a complete orthonormalized set.

Define integration measure in $SH(t1, t2)$ by the natural integration measure associated with the metric of $SH(t1, t2)$ obtained by inducing the metric of SH to its submanifold $SH(t1, t2)$. Denote this integration measure by the symbol

$$DX^3(t1, t2) \tag{89a}$$

Define the integration measure over the theta parameters as the product of the theta parameter integration measures associated with manifolds with topologies $t1$ and $t2$ multiplied by delta functions forcing the various constraints between theta parameters; these linear constraints were derived in Section 4 [see formulas (65)-(70)]. Denote this integration measure by the symbol

$$D\theta(t1, t2)D\bar{\theta}(t1, t2) \tag{89b}$$

Furthermore, denote the product of these integration measures by the symbol

$$DV(t1, t2) = DX^3(t1, t2)D\theta(t1, t2)D\bar{\theta}(t1, t2) \tag{89c}$$

With these preliminaries we are ready to derive a formal solution to the continuity conditions. The continuity conditions state that for two "neighboring" topologies ti, $i=1,2$, the orthonormalized state functionals belonging to $V(t1)$ [$V(t2)$] are expressible as a linear combination of the corresponding state functionals belonging to $V(t2)$ [$V(t1)$]:

$$S^m(ti) = \sum_n S^n(tj)G^{nm}(tj, ti) \qquad (i \neq j) \tag{90a}$$

These conditions can be expressed in a more concise form using matrix notation

$$S(ti) = S(tj)G(tj, ti) \qquad (i \neq j) \tag{90b}$$

The components of the matrices $G(t1, t2)$ and $G(t2, t1)$ are the unknown quantities we wish to solve.

The matrices $G(ti, tj)$ and $G(tj, ti)$ are inverse matrices in the sense that the following equations hold true:

$$G(ti, tj)G(tj, ti) = Id(ti) \tag{91}$$

In order to solve the components of the unknown matrices form the continuity conditions (89) we multiply them with a given state functional $\bar{S}^m(ti)$ of $V(ti)$ and perform the integral over theta parameters and over $SH(t1, t2)$ (overlap integral over singular manifolds). Defining the matrices $H(ti, tj)$ $(i, j = 1, 2)$ by the following formula,

$$\overset{mn}{H}(ti, tj) = (S^m(ti), S^n(tj)) = \int \bar{S}^m(ti)S^n(tj)DV(t1, t2) \tag{92}$$

one can cast the continuity conditions in the following form:

$$H(tj, ti) = H(ti, tk)G(tk, ti) \tag{93}$$

Using the equations (91a) it is easy to verify that only two of these equations are independent of each other. For example, the equations corresponding to index pairs $(i, j) = (1, 1)$ and (38) imply the remaining equations.

If the matrix $H(ti, tj)$ is invertible in the sense that there exists a matrix $I(tj, ti)$ with the property

$$I(ti, tj)H(tj, ti) = Id(ti) \tag{94}$$

then the unknown matrix $G(ti, tj)$ can be solved from (93) and written in one of the following forms:

$$G(tj, ti) = I(tj, tk)H(tk, ti) \tag{95}$$

Thus we have expressed the matrices $G(ti, tj)$ in terms of overlap integrals of the bare state functionals over the set of singular manifolds, which are in principle calculable.

The assumption about the invertibility of the matrix $H(ti, ti)$ is clearly a crucial step in the formal solution of the continuity conditions. The following argument based on a finite-dimensional analogy indeed shows that the invertibility requirement makes sense.

(1) The solutions of the massless d'Alambertian associated with a finite-dimensional space H form a complete set when restricted to the boundary of a suitable open submanifold of the space H. Examples are as follows:

(i) The solutions of massless d'Alambertian in M^4 form a complete set when restricted to a spacelike hyperplane.

(ii) The solutions of Laplace equation in the 3-ball of a given radius form a complete set when restricted to the surface of the ball.

(2) If this phenomenon occurs also in the infinite-dimensional case one expects that the solutions of d'Alambert-type equations when restricted to a boundary of a suitable open submanifold of the space in question form a complete set.

(3) The sets $SH(t1, t2)$ do belong to the boundary of $SH(t)$ and thus the restrictions of $SH(ti)$ state functionals might well form a complete set in $SH(t1, t2)$.

6.3. One-Valuedness Requirement

The continuation of the bare n-particle state function $S^m(t)$ restricted to $SH(t)$ to a stationary state functional $S_s^m(t)$ (having no sharp particle number) can be performed by applying the formal solution of the continuity conditions. Thus the stationary state functional can be written as a sum of bare state functionals

$$S_s^m(t) = S^m(t) + \sum_{j,n} S^n(tj) G^{nm}(tj, t) \tag{96}$$

Here the matrices $G(t, tj)$ can be decomposed into products of the matrices $G(t, tj)$ associated with the continuations between "neighboring" 3-manifold topologies (there exists 3-surfaces having topology intermediate between to 3-manifold topologies):

$$G(tj, t) = G(tj, t1) G(t1, t2) \cdots G(tm, t) \tag{97}$$

In general it is kinematically possible (the intermediate states in the continuation are "on mass shell states") perform the continuation $t \to tj$ via several paths and each of these paths must lead to the same final result. The uniqueness of the final result is guaranteed if the product of the matrices $G(tj, t)$ associated with a given path of continuation depends only on the initial and final topologies. Equivalently, the product of matrices G associated with a closed kinematically allowed path of continuations $t \to tm \to \cdots t1 \to t$ is always a unit matrix:

$$G(t, t1)G(t1, t2) \cdots G(tm, t) = Id(t) \tag{98}$$

It should be emphasized that the continuations are strongly restricted by the kinematical constraints since the intermediate states of the continuation must be solutions of the field equations: thus all particles in intermediate states must be on mass shell particles. It might well happen that one-valuedness conditions pose strong restrictions on the allowed bare particle states (the impossibility of continuing the free quark state functionals to one-valued state functionals in SH implies the nonobservability of free quarks?!).

One can represent the various continuations diagrammatically. The diagrammatic rules are the following:

(i) Associate with each connected 3-manifold a line with labels describing the topology of the 3-manifold and various quantum numbers of the corresponding bare state functionals.

(ii) The particle number changing transitions have as the basic vertex the 3-particle vertex and the vertex is described by the matrix G.

(iii) The vertices changing 3-manifold topology but preserving connectedness are described by a two-particle vertex described by the matrix G.

In this manner one can associate a diagrammatic representation with each continuation via intermediate topologies.

These diagrams differ from Feynmann diagrams in several respects.

(i) The one-valuedness conditions state that all diagrams having same initial and final states are equivalent so that any reaction can be described by a unique minimal diagram.

(ii) The particles appearing in intermediate lines are on "massshell particles" and therefore all topologically allowed diagrams are not possible kinematically. Thus the one-valuedness conditions are not so stringent as one might first think.

The diagrammatic representation for $2 \to 2$ reaction reveals that the one-valuedness conditions are analogous to the duality conditions familiar from the dual string models (Jacob, 1974; Chew and Rosensweig, 1978; Schwartz, 1985) stating that the sum over the resonances in the s-channel is equivalent to the sum over the exchanges in the t channel.

It should be emphasized, however, that the one-valuedness conditions imply restrictions on the transition amplitudes only when both s- and t-channel reactions can proceed on shell. The assumption about crossing symmetry for $2 \to 2$-channel reactions might imply the duality conditions in their full strength. Motivated by the analogy with dual models we shall in the sequel refer to the one-valuedness conditions as generalized duality conditions.

The physical content of the generalized duality conditions is that one can regard all reaction mechanisms connecting given bare many-particle states to each other as physically equivalent. The predictions deriving from the generalized duality might serve as direct tests of the theory.

6.4. Construction of the S Matrix

Since the relationship between bare and stationary states resembles the relationship between incoming and outgoing states in field theories it seems natural to define the S matrix as the matrix relating these two sets of sttates to each other.

Whenever possible we shall use the shorthand notations $|m)$ and $|m_s)$ for the bare and stationary states, respectively. The bare states are assumed to be orthonormalized with respect to the scalar product, whose existence follows from the phase symmetry of the action. The scalar product is assumed to be positive definite:

$$(m, n) = \delta_{m,n} \tag{99}$$

The stationary states are not expected to be orthogonal as such and the scalar products between stationary states can be represented in a form of a matrix,

$$(m_s, n_s) = (Id + G + G^+ + G^+G)_{m,n} \tag{100a}$$

Here we have used the following notations:

$$Id = \sum_t Id(t) \tag{100b}$$

$$G = \sum_{ti,tj} G(ti, tj) \tag{100c}$$

$$G^+ = \sum_{ti,tj} G^+(ti, tj) \tag{100d}$$

Since the matrix formed by the scalar products is Hermitian it is possible to perform a unitary transformation U making this matrix diagonal. It is clear that the diagonalizing transformation mixes stationary states corresponding to different topologies. We assume however that the mixing is so small that there exists a natural correspondence between the bare states $|m)$ and the new diagonalized states $|\tilde{m}_s)$. The diagonalization is necessary in order to define positive definite transition probabilities.

In the orthogonalized basis the matrix formed by the scalar products has the form

$$(\tilde{m}_s, \tilde{n}_s) = Z(m)\delta_{m,n} \tag{101}$$

Here the constants $Z(m)$ are analogous to the wave function renormalization constants of the ordinary quantum field theories.

With these preliminaries we are ready to define the S matrix and its dual via the following formula:

$$S_{nm} = (n, \tilde{m}_s)/[Z(m)]^{1/2} \tag{102}$$

The unitarity of the S matrix and thus the existence of positive definite transition probabilities follows from the assumption that the scalar products between the bare states are positive definite.

In the orthogonalized basis the representations of the S matrix in terms of the matrix G is given by the formula

$$S_{nm} = A_{nm}/[Z(m)]^{1/2} \tag{103a}$$

$$A = (Id + G)U \tag{103b}$$

It is rather easy to demonstrate that the nontriviality of the matrix U is necessary in order to obtain a physically acceptable S matrix. Assuming that the U matrix is the identity matrix, the S-matrix elements between state functionals $S^m(t)$ and $S^n(t)$ are diagonal:

$$(S^m(t), S^n(t)) = \delta_{m,n} \tag{104}$$

Thus the S-matrix would be nontrivial only for topology changing transition. For instance, for the scattering of two charged particles the S matrix would be trivial!

The mixing of different 3-topologies caused by the matrix U is necessary in order to explain Cabibbo mixing in the TGD framework. If different fermion families correspond to different boundary component topologies of a some 3-manifold having one boundary component then Cabibbo mixing can be identified as a mixing of different boundary topologies caused by the diagonalizing matrix U.

7. SUMMARY AND OUTLOOK

The construction of a dynamic theory based on the basic ideas of TGD represented in earlier papers has been our principal object of interest during several years. The attempts to construct a dynamical theory based on the direct generalization of the functional integral formalism have had a limited success; this approach has produced only semiclassical arguments.

In this paper we have adopted a new quantization philosophy based on the belief that the idea of constructing a quantum theory by quantizing a classical theory is the real source of difficulties. A concise representation of the new "quantization philosophy" is the statement "Do not quantize!"

Our attempt to realize this Philosophy relies on the observation that the ordinary Schrödinger equation is derivable from the action principle and the existence of the S matrix follows from the phase symmetry of the action.

This observation leads to the idea that quantum dynamics is defined by a "super d'Alambert equation" (states with arbitrary fermion number correspond to the solutions of the super d'Alambertian), derivable from a quadratic variational principle, in the sapce SH of the space like 3-submanifolds of the space H.

A nice feature of this approach is that it predicts "everything." Since "massless" d'Alambert-type operators contain no arbitrary dimensionless parameters (linearity!) all dimensionless coupling constants must be predictions of the theory. Furthermore, all dimensional parameters must be related to the length scale defined by the size of the space CP_2.

A second nice feature of this approach is that it takes the equivalence principle to its extreme; one can regard quantum dynamics simply as a classical free field theory with phase symmetry defined in an appropriate configuration space.

The nontriviality of the theory follows from uncertainty principle in space SH. Any state functional having a sharp particle number, that is, restricted to a submanifold of SH consisting of 3-manifolds with fixed number of components with given topologies, is nonstationary and begins to disperse to other parts of SH with different particle numbers. This dispersion is observed as various particle number changing reactions.

The technical realization of these ideas is attempted and proceeds in the following steps:

(1) The geometrization of the space SH. This means defining of the metric, vielbein, and spinor structures in SH. These structures are induced from the corresponding structures of the space H.

(2) Definition of the superfield concept. This is accomplished by introducing the spinor Grassmann algebra as an algebra generated by the "theta parameters" in one to one correspondence with a spinor basis in point of SH. A scalar superfield is defined as a field having values in spinor Grassmann algebra.

(3) Definition of the super d'Alambertian. The concept is first defined in the finite-dimensional case and then generalized to the actual case of interest. It is found that the requirement of the so-called super gauge invariance leads to a unique super d'Alambertian in the general case.

In the case of $M^4 \times CP_2$ the Kähler structure of CP_2 makes possible the introduction of a CP-breaking term in the super d'Alambertian but it turns out that the requirement of a maximal symmetry for the "pointlike limit" of the theory (super d'Alambertian in H) fixes the CP-breaking term uniquely. The super d'Alambertian of H has (in a certain sense local) supersymmetry as a dynamical symmetry.

Furthermore, it seems necessary to pose the chirality condition on the superfield.

In the constructionn of the S matrix the concepts of bare and stationary state play central role. Bare states are superscalar fields restricted to an open subset of SH consisting of 3-manifolds with a given number of components with given topologies and are stationary solutions only in this subset. Stationary states are obtained by continuing the bare state functionals to state functionals defined in whole SH.

The continuity conditions making it possible to continue a bare state functional to whole SH can be solved formally and the conditions guaranteeing the uniqueness of the continuation process turn out to be analogous to the conditions defining the duality concept in the context of dual models.

The S matrix can be defined as a matrix relating to each other the bare and the stationary states. The S matrix is unitary provided the scalar product associated with the super d'Alambert equation is positive definite; this requirement might lead to strong constraints concerning the choice of the space H. The calculation of the S-matrix elements reduces to the solution of the continuity conditions. Since the basis of the stationary states is not normalized to unity, quantities analogous to the wave function renormalization constants appear in the expressions for the transition probabilities.

We believe that these long-waited successes at the formal level suffice to motivate future efforts to develop the theory. Besides developing the formalism there are indeed several problems to be studied. We mention only some of the most important:

(i) Define and study the classical limit of the theory (classical orbit the 3-surface four surface extremizing some effective action?).

(ii) Study the implications of the characteristic features of the S-matrix (intermediate particles are "on mass shell" particles; generalized duality).

(iii) Try to relate the proposed formalism to functional integral formalism.

(iv) Study the properties of the CP-breaking super d'Alambertian in H (this might have something to do with the point particle limit of the theory).

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APPENDIX: THE EXISTENCE OF THE S MATRIX AND THE PHASE SYMMETRY OF THE VARIATIONAL PRINCIPLE ASSOCIATED WITH THE SCHRÖDINGER EQUATION

The Schrödinger equation for a particle moving in the potential $V(x)$ is of the form

$$OP\psi = 0 \tag{A1a}$$

$$OP = (i)_{+} + \nabla^2 + V) \tag{A1b}$$

This equation can be derived from the following variational principle:

$$S = \int L\, d^3x\, dt \tag{A2a}$$

$$L = \bar{\psi} OP\psi \tag{A2b}$$

The phase symmetry

$$\psi \to \exp(i\alpha)\psi \tag{A3}$$

of the action implies the conservation of probability p defined as

$$p = \int \bar{\psi}\psi\, d^3x \tag{A4}$$

Let the set

$$\{\psi^m(x)\} \tag{A5}$$

form a complete orthonormalized basis of state functions at time $t = 0$ and let $\psi^m(t)$ be a state function satisfying the condition

$$\psi^m(x, 0) = \psi^m(x) \tag{A6}$$

From the linearity of the Schrödinger equation it follows that (t) can be expressed in the form

$$\psi^m(x, t) = \sum_n S^{mn}(t)\psi^n(x) \tag{A7}$$

The requirement of probability conservation (A4) implies that the matrix S satisfies the unitary condition

$$SS^+ = S^+S = Id \tag{A8}$$

Thus the existence of a unitary S-matrix follows from phase symmetry.

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